

Available online at [www.sciencedirect.com](http://www.sciencedirect.com) **ScienceDirect**

Journal of Algebra 319 (2008) 3053–3061

---

**JOURNAL OF  
Algebra**

---

[www.elsevier.com/locate/jalgebra](http://www.elsevier.com/locate/jalgebra)

# Recollements, idempotent completions and t-structures of triangulated categories <sup>☆</sup>

Qinghua Chen <sup>a</sup>, Lidan Tang <sup>b,\*</sup><sup>a</sup> *School of Mathematics and Computer Science, Fujian Normal University, Fuzhou 350007, PR China*<sup>b</sup> *School of Mathematical Science, Xiamen University, Xiamen 361005, PR China*

Received 16 May 2007

Available online 3 December 2007

Communicated by Michel Van den Bergh

---

## Abstract

We first prove that the idempotent completion of a right or left recollement of triangulated categories is still a right or left recollement, then show that the t-structure on a triangulated category is compatible with taking idempotent completion. Finally, an application of the main theorem is given, which is focused on the boundedness and nondegeneration of the t-structure induced by a recollement and its idempotent completion.

© 2007 Elsevier Inc. All rights reserved.

**Keywords:** Right recollement; Left recollement; Idempotent completion; t-Structure

---

## 1. Introduction

The notion of recollement of triangulated categories was introduced by Beilinson, Bernstein and Deligne in connection with derived categories of sheaves on topological spaces with the idea that one such category as being “glued together” from two others [2].

---

<sup>☆</sup> This work has been supported by the National Natural Science Foundation of China (10671161), the Natural Science Foundation of Fujian Province (Z0511022) and the Foundation of Education Committee of Fujian Province (JA050206 and JB04251).

\* Corresponding author.

*E-mail addresses:* [cqhmath@fjnu.edu.cn](mailto:cqhmath@fjnu.edu.cn) (Q. Chen), [lidantang@126.com](mailto:lidantang@126.com) (L. Tang).

**Definition 1.** (See Beilinson, Bernstein and Deligne [2].) Let  $\mathcal{D}$ ,  $\mathcal{D}'$ ,  $\mathcal{D}''$  be triangulated categories. Then a recollement of  $\mathcal{D}$  relative to  $\mathcal{D}'$  and  $\mathcal{D}''$ , diagrammatically expressed by

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\ \mathcal{D}' & \xrightarrow{i_* = i_!} & \mathcal{D} & \xrightarrow{j^! = j^*} & \mathcal{D}'' \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array}$$

given by six exact functors  $i^*$ ,  $i_* = i_!$ ,  $i^!$ ,  $j_!$ ,  $j^! = j^*$ ,  $j_*$  satisfy the following four conditions:

- (R1)  $(i^*, i_* = i_!, i^!)$  and  $(j_!, j^! = j^*, j_*)$  are adjoint triples, i.e.,  $i^*$  is left adjoint to  $i_*$  which is left adjoint to  $i^!$ , etc.;
- (R2)  $i^! j_* = 0$ ;
- (R3)  $i_*$ ,  $j_!$ ,  $j_*$  are full embeddings;
- (R4) any object  $X$  in  $\mathcal{D}$  determines triangles

$$i_* i^! X \rightarrow X \rightarrow j_* j^! X \rightarrow T(i_* i^! X) \quad \text{and} \quad j_! j^* X \rightarrow X \rightarrow i_! i^* X \rightarrow T(j_! j^* X)$$

where the morphisms  $i_* i^! X \rightarrow X$ ,  $X \rightarrow j_* j^! X$ ,  $j_! j^* X \rightarrow X$  and  $X \rightarrow i_! i^* X$  are the adjunction morphisms.

A prototypical example of a recollement is as follows which is given by Beilinson, Bernstein and Deligne in [2]:

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\ D(Z) & \xrightarrow{i_* = i_!} & D(X) & \xrightarrow{j^! = j^*} & D(U) \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array}$$

is a recollement of  $D(X)$  relative to  $D(Z)$  and  $D(U)$ , where  $X$  is a topological space equal to the union of the closed subset  $Z$  and the open complement  $U$  and  $D(Z)$ ,  $D(X)$ ,  $D(U)$  are suitable derived categories of sheaves.

Two weaker forms of recollements are introduced by B. Parshall:

A right recollement is said to hold if the lower two rows of a recollement (as defined above) exist and the functors appearing in these two rows (i.e.,  $i_*$ ,  $i^!$ ,  $j^!$  and  $j_*$ )

$$\begin{array}{ccccc} & \xrightarrow{i_* = i_!} & & \xrightarrow{j^! = j^*} & \\ \mathcal{D}' & & \mathcal{D} & & \mathcal{D}'' \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array}$$

satisfy all the conditions in the definition above which involve only these functors.

A left recollement is defined via the upper two rows similarly.

**Definition 2.** (See [1,5].) Let  $\mathcal{A}$  be an additive category, an idempotent morphism  $e : A \rightarrow A$  is said to split if there are two morphisms  $p : A \rightarrow B$  and  $q : B \rightarrow A$  such that  $e = q \circ p$  and  $p \circ q = \text{Id}_B$ .

The additive category  $\mathcal{A}$  is said to be idempotent complete (or Karoubian) provided that every idempotent morphism splits.

Many natural triangulated categories, such as the derived categories of perfect complexes over a quasi-separated, quasi-compact scheme and the bounded derived categories of abelian categories, are idempotent complete. But not all the triangulated categories are idempotent complete. Fortunately, any additive category can be idempotent completed and the idempotent completion of a triangulated category is still a triangulated category [1].

**Definition 3.** (See [1].) Let  $\mathcal{A}$  be an additive category. The idempotent completion (or Karoubianisation) of  $\mathcal{A}$  is the category  $\tilde{\mathcal{A}}$  defined as follows. Objects of  $\tilde{\mathcal{A}}$  are pairs  $(A, e)$ , where  $A \in \mathcal{A}$  and  $e \in \text{Hom}_{\mathcal{A}}(A, A)$  is an idempotent. A morphism  $\alpha \in \text{Hom}_{\tilde{\mathcal{A}}}((A, e), (B, f))$  is a morphism  $\alpha \in \text{Hom}_{\mathcal{A}}(A, B)$  such that  $\alpha \circ e = \alpha = f \circ \alpha$ .

Note that the assignment  $A \rightarrow (A, \text{Id}_A)$  defines a functor  $\iota_{\mathcal{A}}$  from  $\mathcal{A}$  to  $\tilde{\mathcal{A}}$ . It is well known that the functor  $\iota_{\mathcal{A}}$  is fully-faithful. Moreover,  $\mathcal{A}$  is idempotent complete if and only if  $\iota_{\mathcal{A}}$  is an equivalence. So from now on we will think of  $\mathcal{A}$  as a full subcategory of  $\tilde{\mathcal{A}}$ .

Let  $\mathcal{D}$  be a triangulated category,  $T$  be its shift functor. Define functor  $\tilde{T}$  as follows

$$\begin{aligned}\tilde{T}: \tilde{\mathcal{D}} &\rightarrow \tilde{\mathcal{D}}, \\ (A, e) &\mapsto (T(A), T(e)), \\ \alpha &\mapsto T(\alpha)\end{aligned}$$

where  $(A, e) \in \tilde{\mathcal{D}}, \alpha \in \text{Hom}_{\tilde{\mathcal{D}}}((A, e), (B, f))$ . A sextuple  $\Delta: (A, e) \xrightarrow{\alpha} (B, f) \xrightarrow{\beta} (C, g) \xrightarrow{\gamma} \tilde{T}(A, e)$  in  $\tilde{\mathcal{D}}$  is said to be a triangle if there exists a sextuple  $\Delta'$  in  $\tilde{\mathcal{D}}$  such that  $\Delta \oplus \Delta'$  is isomorphic to a triangle in  $\mathcal{D}$ . It is shown in [1] that with shift functor  $\tilde{T}$  and triangles in  $\tilde{\mathcal{D}}$  defined above,  $\tilde{\mathcal{D}}$  is a triangulated category.

Our main theorem is

**Theorem 4.** Let  $\mathcal{D}, \mathcal{D}', \mathcal{D}''$  be triangulated categories. Assume that  $\mathcal{D}$  admits a right recollement relative to triangulated categories  $\mathcal{D}'$  and  $\mathcal{D}''$ , that is

$$\mathcal{D}' \begin{array}{c} \xrightarrow{i_* = i_!} \\ \xleftarrow{i^!} \end{array} \mathcal{D} \begin{array}{c} \xrightarrow{j^! = j^*} \\ \xleftarrow{j_*} \end{array} \mathcal{D}''.$$

Then  $\tilde{\mathcal{D}}$  admits a right recollement relative to  $\tilde{\mathcal{D}}'$  and  $\tilde{\mathcal{D}}''$  as follows

$$\tilde{\mathcal{D}}' \begin{array}{c} \xrightarrow{\tilde{i}_* = \tilde{i}_!} \\ \xleftarrow{\tilde{i}^!} \end{array} \tilde{\mathcal{D}} \begin{array}{c} \xrightarrow{\tilde{j}^! = \tilde{j}^*} \\ \xleftarrow{\tilde{j}_*} \end{array} \tilde{\mathcal{D}}''.$$

It is shown in [1] that if  $\mathcal{A}$  is an exact category, then  $\widetilde{D^b(\mathcal{A})} \cong D^b(\tilde{\mathcal{A}})$ . So we have

**Corollary 5.** Suppose that  $\mathcal{A}, \mathcal{A}', \mathcal{A}''$  are exact categories. If a triangulated category  $D^b(\mathcal{A})$  admits a right recollement relative to  $D^b(\mathcal{A}')$  and  $D^b(\mathcal{A}'')$ , then  $D^b(\tilde{\mathcal{A}})$  admits a right recollement relative to  $D^b(\tilde{\mathcal{A}}')$  and  $D^b(\tilde{\mathcal{A}}'')$ .

**Remark 6.** All of the results above are also true for left recollements and recollements.

Suppose that a triangulated category  $\mathcal{D}$  admits a recollement relative to  $\mathcal{D}'$  and  $\mathcal{D}''$ . If  $\mathcal{D}'$ ,  $\mathcal{D}''$  are idempotent complete, then  $\iota_{\mathcal{D}'}$ ,  $\iota_{\mathcal{D}''}$  are equivalences. Since  $(\iota_{\mathcal{D}'}, \iota_{\mathcal{D}}, \iota_{\mathcal{D}''})$  is a comparison functor between two recollements

$$\begin{array}{ccccc}
 \mathcal{D}' & \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_* = i_!} \\ \xleftarrow{i^!} \end{array} & \mathcal{D} & \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^! = j^*} \\ \xleftarrow{j_*} \end{array} & \mathcal{D}'' \\
 \downarrow \iota_{\mathcal{D}'} & & \downarrow \iota_{\mathcal{D}} & & \downarrow \iota_{\mathcal{D}''} \\
 \widetilde{\mathcal{D}'} & \begin{array}{c} \xleftarrow{\widetilde{i}^*} \\ \xrightarrow{\widetilde{i}_* = \widetilde{i}_!} \\ \xleftarrow{\widetilde{i}^!} \end{array} & \widetilde{\mathcal{D}} & \begin{array}{c} \xleftarrow{\widetilde{j}_!} \\ \xrightarrow{\widetilde{j}^! = \widetilde{j}^*} \\ \xleftarrow{\widetilde{j}_*} \end{array} & \widetilde{\mathcal{D}''}
 \end{array}$$

according to Theorem 2.5 in [6],  $\iota_{\mathcal{D}}$  is an equivalence, which implies that  $\mathcal{D}$  is idempotent complete. Conversely, both  $\mathcal{D}'$  and  $\mathcal{D}''$  are thick inside  $\mathcal{D}$  by [3]. Note that any thick subcategory of an idempotent complete category is still idempotent complete. We have that  $\mathcal{D}'$  and  $\mathcal{D}''$  are idempotent complete. So we have

**Corollary 7.** *Let  $\mathcal{D}$ ,  $\mathcal{D}'$ ,  $\mathcal{D}''$  be triangulated categories. Assume that  $\mathcal{D}$  admits a recollement relative to  $\mathcal{D}'$  and  $\mathcal{D}''$ . Then  $\mathcal{D}$  is idempotent complete if and only if  $\mathcal{D}'$ ,  $\mathcal{D}''$  are idempotent complete.*

It is well known that an additive category is Krull–Schmidt if and only if it is idempotent complete and for each object  $X$ ,  $\text{End}(X)$  is a semiperfect ring [4]. For a recollement, since  $i_*$ ,  $j_*$ ,  $j_!$  are full embeddings, it follows that

**Corollary 8.** *Let  $\mathcal{D}$ ,  $\mathcal{D}'$ ,  $\mathcal{D}''$  be triangulated categories. Let  $\mathcal{D}$  admit a recollement relative to  $\mathcal{D}'$  and  $\mathcal{D}''$ . If  $\mathcal{D}$  is a Krull–Schmidt category, then  $\mathcal{D}'$ ,  $\mathcal{D}''$  are Krull–Schmidt categories.*

In this paper, Section 2 is due to the proof of the main theorem.

In Section 3, we first prove that t-structures on triangulated categories are compatible with taking idempotent completions (Theorem 15), then give an application of the main theorem on t-structures (Proposition 17).

## 2. Proof of theorem

Before proving the theorem, we need some preparations.

Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an additive functor between two additive categories. Then the assignment  $(A, e) \mapsto (F(A), F(e))$  defines an additive functor  $\widetilde{F}$  from  $\widetilde{\mathcal{A}}$  to  $\widetilde{\mathcal{B}}$  satisfying  $\iota_{\mathcal{B}} \circ F = \widetilde{F} \circ \iota_{\mathcal{A}}$ .

Moreover, given two additive functors  $F, G : \mathcal{A} \rightarrow \mathcal{B}$  and a natural transformation  $\delta : F \rightarrow G$ . Then there exists a natural transformation  $\widetilde{\delta} : \widetilde{F} \rightarrow \widetilde{G}$  satisfying  $\widetilde{\delta}_{(A, e)} = G(e) \circ \delta_A \circ F(e)$ , for arbitrary  $(A, e) \in \mathcal{A}$ .

**Lemma 9.** *Let  $F : \mathcal{D} \rightarrow \mathcal{D}'$ ,  $G : \mathcal{D}' \rightarrow \mathcal{D}$  be exact functors between two triangulated categories. If  $(F, G)$  is an adjoint pair, then  $(\widetilde{F}, \widetilde{G})$  is an exact adjoint pair.*

**Proof.** For any triangle  $\Delta : (A, e) \xrightarrow{\alpha} (B, f) \xrightarrow{\beta} (C, g) \xrightarrow{\gamma} \widetilde{T}(A, e)$  in  $\widetilde{\mathcal{D}}$ , there exists a triangle  $\Delta'$  in  $\widetilde{\mathcal{D}}$  such that  $\Delta \oplus \Delta'$  is isomorphic to a triangle in  $\mathcal{D}$ . Since  $F$  is exact, it follows that

$F(\Delta \oplus \Delta') \cong \tilde{F}(\Delta) \oplus \tilde{F}(\Delta')$  is isomorphic to a triangle in  $\mathcal{D}'$ . Thus,  $\tilde{F}(\Delta)$  is a triangle in  $\tilde{\mathcal{D}}'$ , which implies that  $\tilde{F}$  is exact. Similarly,  $\tilde{G}$  is an exact functor.

Note that  $(F, G)$  is an adjoint pair, there exists a natural isomorphism

$$\eta_{A,A'} : \text{Hom}_{\mathcal{D}'}(FA, A') \rightarrow \text{Hom}_{\mathcal{D}}(A, GA')$$

for any  $A \in \mathcal{D}$ ,  $A' \in \mathcal{D}'$ . Then,  $\eta_{A,A'}$  induces a natural isomorphism between  $\text{Hom}_{\tilde{\mathcal{D}}'}(\tilde{F}((A, e)), (A', e'))$  and  $\text{Hom}_{\tilde{\mathcal{D}}}((A, e), \tilde{G}((A', e')))$  for each  $(A, e) \in \tilde{\mathcal{D}}$ ,  $(A', e') \in \tilde{\mathcal{D}}'$ . In fact, let  $\alpha \in \text{Hom}_{\tilde{\mathcal{D}}'}(\tilde{F}((A, e)), (A', e'))$ . Then  $\alpha \circ Fe = \alpha = e' \circ \alpha$ , which implies that  $\eta_{A,A'}(\alpha) \circ e = \eta_{A,A'}(\alpha) = Ge' \circ \eta_{A,A'}(\alpha)$ . That means  $\eta_{A,A'}(\alpha) \in \text{Hom}_{\tilde{\mathcal{D}}}((A, e), \tilde{G}((A', e')))$ . On the other hand, let  $\alpha \in \text{Hom}_{\mathcal{D}'}(FA, A')$  and  $\eta(\alpha) \in \text{Hom}_{\tilde{\mathcal{D}}}((A, e), \tilde{G}((A', e')))$ . Then  $\eta_{A,A'}(\alpha) \circ e = \eta_{A,A'}(\alpha) = Ge' \circ \eta_{A,A'}(\alpha)$ , which implies that  $\alpha \circ Fe = \alpha = e' \circ \alpha$ . That is  $\alpha \in \text{Hom}_{\tilde{\mathcal{D}}'}(\tilde{F}((A, e)), (A', e'))$ . And it is easy to see that  $\eta$  is natural.

Thus,  $(\tilde{F}, \tilde{G})$  is an exact adjoint pair.  $\square$

**Remark 10.** If  $(F, G)$  is an adjoint pair, then there exist two adjunction natural transformations  $\delta : FG \rightarrow \text{Id}_{\mathcal{B}}$  and  $\varepsilon : \text{Id}_{\mathcal{A}} \rightarrow GF$ . Meanwhile, by Lemma 9, there exist two adjunction natural transformations  $\delta' : \tilde{F}\tilde{G} \rightarrow \text{Id}_{\tilde{\mathcal{B}}}$  and  $\varepsilon' : \text{Id}_{\tilde{\mathcal{A}}} \rightarrow \tilde{G}\tilde{F}$ . It is easy to see that  $\tilde{\delta} = \delta'$  and  $\tilde{\varepsilon} = \varepsilon'$ .

**Lemma 11.** Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an additive functor,  $\mathcal{A}' \subseteq \mathcal{A}$  a full subcategory such that every object in  $\mathcal{A}$  is a direct summand of an object in  $\mathcal{A}'$ . If  $F|_{\mathcal{A}'}$  is fully-faithful, then  $F$  is fully-faithful.

We have the following key observation by the naturalness of the natural transformation  $\delta$ .

**Lemma 12.** Let  $\mathcal{A}, \mathcal{B}$  be two additive categories and  $A \in \mathcal{A}$ . For each idempotent morphism  $e \in \text{Hom}_{\mathcal{A}}(A, A)$ , there exists a natural isomorphism in  $\tilde{\mathcal{A}}$

$$\begin{pmatrix} e \\ \text{Id}_A - e \end{pmatrix} : A \rightarrow (A, e) \oplus (A, \text{Id}_A - e).$$

Moreover, given two additive functors  $F, G : \mathcal{A} \rightarrow \mathcal{B}$  and a natural transformation  $\delta : F \rightarrow G$ , the following diagram is commutative:

$$\begin{array}{ccc} F(A) & \xrightarrow{\delta_A} & G(A) \\ \downarrow & & \downarrow \\ \tilde{F}((A, e)) \oplus \tilde{F}((A, \text{Id}_A - e)) & \xrightarrow{\begin{pmatrix} \tilde{\delta}_{(A, e)} & 0 \\ 0 & \tilde{\delta}_{(A, \text{Id}_A - e)} \end{pmatrix}} & \tilde{G}((A, e)) \oplus \tilde{G}((A, \text{Id}_A - e)) \end{array}$$

where the vertical morphisms are the natural isomorphisms given above.

**Proof.** It is easy to see that

$$\begin{aligned} \begin{pmatrix} e \\ \text{Id}_A - e \end{pmatrix} \circ (e \quad \text{Id}_A - e) &= \text{Id}_{(A, e) \oplus (A, \text{Id}_A - e)}, \\ (e \quad \text{Id}_A - e) \circ \begin{pmatrix} e \\ \text{Id}_A - e \end{pmatrix} &= \text{Id}_A. \end{aligned}$$

Thus,  $(\text{Id}_A - e)^e$  is an isomorphism.

Moreover, by the naturalness of  $\delta$ , the diagram above is commutative.  $\square$

#### Proof of Theorem 4.

- (R1) By Lemma 9,  $(\tilde{i}_* = \tilde{i}!, \tilde{i}!), (\tilde{j}^! = \tilde{j}^*, \tilde{j}_*)$  are exact adjoint pairs.  
 (R2) Since  $i^! j_* = 0$ , it follows that  $\tilde{i}^! \tilde{j}_* = i^! j_* = 0$ .  
 (R3) Note that  $\mathcal{D}'$  is a full subcategory of  $\mathcal{D}$ , and every object of  $\mathcal{D}'$  is a direct summand of an object in  $\mathcal{D}$ , moreover, the restriction of  $\tilde{i}_*$  is just  $i_*$ . Applying Lemma 11, we deduce that  $\tilde{i}_*$  is fully-faithful. That is,  $\tilde{i}_*$  is a full embedding.  
 (R4) For any  $(A, e) \in \mathcal{D}$ , there exists a triangle in  $\mathcal{D}$  as follows

$$i_* i^! A \xrightarrow{\delta_A} A \xrightarrow{\varepsilon_A} j_* j^! A \xrightarrow{\xi_A} T i_* i^! A$$

where  $\delta_A$  and  $\varepsilon_A$  are adjunction morphisms. We claim that the following diagram is commutative, where  $A_1 = (A, e)$ ,  $A_2 = (A, \text{Id}_A - e)$  and the vertical morphisms are natural isomorphisms describe in Lemma 12.

$$\begin{array}{ccccccc} i_* i^! A & \xrightarrow{\delta_A} & A & \xrightarrow{\varepsilon_A} & j_* j^! A & \xrightarrow{\xi_A} & T i_* i^! A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tilde{i}_* \tilde{i}^! A_1 \oplus \tilde{i}_* \tilde{i}^! A_2 & \xrightarrow{\begin{pmatrix} \tilde{\delta}_{A_1} & 0 \\ 0 & \tilde{\delta}_{A_2} \end{pmatrix}} & A_1 \oplus A_2 & \xrightarrow{\begin{pmatrix} \tilde{\varepsilon}_{A_1} & 0 \\ 0 & \tilde{\varepsilon}_{A_2} \end{pmatrix}} & \tilde{j}_* \tilde{j}^! A_1 \oplus \tilde{j}_* \tilde{j}^! A_2 & \xrightarrow{\begin{pmatrix} \tilde{\xi}_{A_1} & 0 \\ 0 & \tilde{\xi}_{A_2} \end{pmatrix}} & \tilde{T} \tilde{i}_* \tilde{i}^! A_1 \oplus \tilde{T} \tilde{i}_* \tilde{i}^! A_2. \end{array}$$

To see this, we only need to show that  $\xi$  is a natural transformation. In fact, for any  $\alpha \in \text{Hom}_{\mathcal{D}}(A, B)$ , there exists  $\beta \in \text{Hom}_{\mathcal{D}}(j_* j^! A, j_* j^! B)$  such that the following diagram is commutative

$$\begin{array}{ccccccc} i_* i^! A & \xrightarrow{\delta_A} & A & \xrightarrow{\varepsilon_A} & j_* j^! A & \xrightarrow{\xi_A} & T i_* i^! A \\ i_* i^! \alpha \downarrow & & \alpha \downarrow & & \beta \downarrow & & T i_* i^! \alpha \downarrow \\ i_* i^! B & \xrightarrow{\delta_B} & B & \xrightarrow{\varepsilon_B} & j_* j^! B & \xrightarrow{\xi_B} & T i_* i^! B. \end{array}$$

Since  $j_*$  is fully-faithful, there exists  $\gamma \in \text{Hom}_{\mathcal{D}''}(j^! A, j^! B)$  such that  $j_*(\gamma) = \beta$ , which means  $j_*(\gamma) \circ \varepsilon_A = \varepsilon_B \circ \alpha$ . Thus  $\eta_{A, j^! B}(\gamma) = \eta_{A, j^! B}(j^! \alpha)$ , where  $\eta_{A, j^! B} : \text{Hom}_{\mathcal{D}''}(j^! A, j^! B) \rightarrow \text{Hom}_{\mathcal{D}}(A, j_* j^! B)$  is a natural isomorphism. So we have  $\gamma = j^! \alpha$ , which implies that  $\xi_B \circ j_* j^! \alpha = T i_* i^! \alpha \circ \xi_A$ .

Hence, we have a triangle  $\tilde{i}_* \tilde{i}^! A_1 \xrightarrow{\tilde{\delta}_{A_1}} A_1 \xrightarrow{\tilde{\varepsilon}_{A_1}} \tilde{j}_* \tilde{j}^! A_1 \xrightarrow{\tilde{\xi}_{A_1}} \tilde{T} \tilde{i}_* \tilde{i}^! A_1$  in  $\tilde{\mathcal{D}}$ , where  $\tilde{\delta}_{A_1}$  and  $\tilde{\varepsilon}_{A_1}$  are adjunction morphisms by Remark 10.

This completes the proof.  $\square$

### 3. Applications to t-structures

Let us recall some related notions in [2].

**Definition 13.** (See [2].) Let  $\mathcal{D}$  be a triangulated category. A t-structure on  $\mathcal{D}$  is a pair of strictly full subcategories  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  satisfying the following conditions: If we put  $\mathcal{D}^{\leq n} = T^{-n}\mathcal{D}^{\leq 0}$ ,  $\mathcal{D}^{\geq n} = T^{-n}\mathcal{D}^{\geq 0}$  for  $n \in \mathbb{Z}$ , we have

- (T1)  $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 1}$ ,  $\mathcal{D}^{\geq 0} \supseteq \mathcal{D}^{\geq 1}$ ;
- (T2)  $\text{Hom}_{\mathcal{D}}(X, Y) = 0$  for  $X \in \mathcal{D}^{\leq 0}$ ,  $Y \in \mathcal{D}^{\geq 1}$ ;
- (T3) for any  $X \in \mathcal{D}$ , there exists a triangle  $X_0 \rightarrow X \rightarrow X_1 \rightarrow TX_0$  such that  $X_0 \in \mathcal{D}^{\leq 0}$ ,  $X_1 \in \mathcal{D}^{\geq 1}$ .

**Definition 14.** (See [2].) Let  $\mathcal{D}$  be a triangulated category. Then a t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is called bounded if  $\bigcup_{n \in \mathbb{Z}} \mathcal{D}^{\leq n} = \mathcal{D}$  and  $\bigcup_{n \in \mathbb{Z}} \mathcal{D}^{\geq n} = \mathcal{D}$ . The t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is called nondegenerate if  $\bigcap_{n \in \mathbb{Z}} \mathcal{D}^{\leq n} = 0$  and  $\bigcap_{n \in \mathbb{Z}} \mathcal{D}^{\geq n} = 0$ .

The following theorem show that t-structures on triangulated categories are compatible with taking idempotent completions.

**Theorem 15.** Let  $\mathcal{D}$  be a triangulated category. Then a t-structure on  $\mathcal{D}$   $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  induces a t-structure  $(\tilde{\mathcal{D}}^{\leq 0}, \tilde{\mathcal{D}}^{\geq 0})$  on  $\tilde{\mathcal{D}}$ , where

$$\text{obj } \tilde{\mathcal{D}}^{\leq 0} = \{(A, e) \mid A \in \mathcal{D}^{\leq 0}, e \in \text{Hom}_{\mathcal{D}}(A, A) \text{ is an idempotent}\},$$

$$\text{obj } \tilde{\mathcal{D}}^{\geq 0} = \{(B, f) \mid B \in \mathcal{D}^{\geq 0}, f \in \text{Hom}_{\mathcal{D}}(B, B) \text{ is an idempotent}\}.$$

**Proof.** (T1) For any  $(A, e)$  in  $\tilde{\mathcal{D}}^{\leq 0}$ , we have  $A \in \mathcal{D}^{\leq 0}$ . Since  $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 1}$ , it follows that  $A \in \mathcal{D}^{\leq 1}$ , which implies that  $\tilde{T}(A, e) = (TA, Te) \in \tilde{\mathcal{D}}^{\leq 0}$ . By definition, we have  $(A, e) \in \tilde{\mathcal{D}}^{\leq 1}$ . Hence,  $\tilde{\mathcal{D}}^{\leq 0} \subseteq \tilde{\mathcal{D}}^{\leq 1}$ . Similarly, we have  $\tilde{\mathcal{D}}^{\geq 0} \supseteq \tilde{\mathcal{D}}^{\geq 1}$ .

(T2) For arbitrary two objects  $(A, e)$  in  $\tilde{\mathcal{D}}^{\leq 0}$  and  $(B, f)$  in  $\tilde{\mathcal{D}}^{\geq 1}$ , we have  $A \in \mathcal{D}^{\leq 0}$ ,  $B \in \mathcal{D}^{\geq 1}$ . Note that  $\text{Hom}_{\tilde{\mathcal{D}}}((A, e), (B, f)) \subseteq \text{Hom}_{\mathcal{D}}(A, B)$  and  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is a t-structure on  $\mathcal{D}$ , by (T2), we have  $\text{Hom}_{\tilde{\mathcal{D}}}((A, e), (B, f)) = 0$ .

(T3) For any  $(A, e) \in \tilde{\mathcal{D}}$ , there exists a triangle  $\tau^{\leq 0}A \xrightarrow{\alpha_A} A \xrightarrow{\beta_A} \tau^{\geq 1}A \xrightarrow{\gamma_A} T\tau^{\leq 0}A$  in  $\mathcal{D}$  by (T3), where  $\tau$ 's are truncation functors. Similar to the proof of Theorem 4, we have the following diagram where the vertical morphisms are isomorphisms explained in Lemma 12 and  $A_1 = (A, e)$ ,  $A_2 = (A, \text{Id}_A - e)$

$$\begin{array}{ccccccc} \tau^{\leq 0}A & \xrightarrow{\alpha_A} & A & \xrightarrow{\beta_A} & \tau^{\geq 1}A & \xrightarrow{\gamma_A} & T\tau^{\leq 0}A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tau^{\leq 0}A_1 \oplus \tau^{\leq 0}A_2 & \xrightarrow{\begin{pmatrix} \tilde{\alpha}_{A_1} & 0 \\ 0 & \tilde{\alpha}_{A_2} \end{pmatrix}} & A_1 \oplus A_2 & \xrightarrow{\begin{pmatrix} \tilde{\beta}_{A_1} & 0 \\ 0 & \tilde{\beta}_{A_2} \end{pmatrix}} & \tau^{\geq 1}A_1 \oplus \tau^{\geq 1}A_2 & \xrightarrow{\begin{pmatrix} \tilde{\gamma}_{A_1} & 0 \\ 0 & \tilde{\gamma}_{A_2} \end{pmatrix}} & \tilde{T}\tau^{\leq 0}A_1 \oplus \tilde{T}\tau^{\leq 0}A_2. \end{array}$$

Thus,  $\tau^{\leq 0}A_1 \xrightarrow{\tilde{\alpha}_{A_1}} A_1 \xrightarrow{\tilde{\beta}_{A_1}} \tau^{\geq 1}A_1 \xrightarrow{\tilde{\gamma}_{A_1}} \tilde{T}\tau^{\leq 0}A_1$  is a triangle in  $\tilde{\mathcal{D}}$ , where  $\tau^{\leq 0}A_1 = (\tau^{\leq 0}A, \tau^{\leq 0}e) \in \tilde{\mathcal{D}}^{\leq 0}$ ,  $\tau^{\geq 1}A_1 = (\tau^{\geq 1}A, \tau^{\geq 1}e) \in \tilde{\mathcal{D}}^{\geq 1}$ .

This completes the proof.  $\square$

**Remark 16.** Let  $\mathcal{D}$  be a triangulated category,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  be a t-structure on  $\mathcal{D}$ . Note that if  $\mathcal{D}$  is idempotent complete, then  $\tilde{\mathcal{D}}^{\leq 0}$  and  $\tilde{\mathcal{D}}^{\geq 0}$  are equivalent to  $\mathcal{D}^{\leq 0}$  and  $\mathcal{D}^{\geq 0}$ .

Let  $\mathcal{D}$ ,  $\mathcal{D}'$ ,  $\mathcal{D}''$  be triangulated categories. Assume  $\mathcal{D}$  admits a recollement relative to  $\mathcal{D}'$  and  $\mathcal{D}''$ . It is shown in [2] that the t-structures  $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$  and  $(\mathcal{D}''^{\leq 0}, \mathcal{D}''^{\geq 0})$  on  $\mathcal{D}'$ ,  $\mathcal{D}''$  can induce a t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}$ , where

$$\begin{aligned}\text{obj } \mathcal{D}^{\leq 0} &= \{X \in \mathcal{D} \mid i^*X \in \mathcal{D}'^{\leq 0} \text{ and } j^!X \in \mathcal{D}''^{\leq 0}\}, \\ \text{obj } \mathcal{D}^{\geq 0} &= \{X \in \mathcal{D} \mid i^!X \in \mathcal{D}'^{\geq 0} \text{ and } j^*X \in \mathcal{D}''^{\geq 0}\}.\end{aligned}$$

Then, by Theorem 15, there is a t-structure  $(\tilde{\mathcal{D}}^{\leq 0}, \tilde{\mathcal{D}}^{\geq 0})$  on  $\tilde{\mathcal{D}}$ . In fact

$$\begin{aligned}\text{obj } \tilde{\mathcal{D}}^{\leq 0} &= \{(X, e) \in \tilde{\mathcal{D}} \mid i^*X \in \mathcal{D}'^{\leq 0} \text{ and } j^!X \in \mathcal{D}''^{\leq 0}\}, \\ \text{obj } \tilde{\mathcal{D}}^{\geq 0} &= \{(X, e) \in \tilde{\mathcal{D}} \mid i^!X \in \mathcal{D}'^{\geq 0} \text{ and } j^*X \in \mathcal{D}''^{\geq 0}\}.\end{aligned}$$

Meanwhile, by Theorem 4,  $\tilde{\mathcal{D}}$  admits a recollement relative to  $\tilde{\mathcal{D}}'$  and  $\tilde{\mathcal{D}}''$ , thus a t-structure  $(\tilde{\mathcal{D}}_1^{\leq 0}, \tilde{\mathcal{D}}_1^{\geq 0})$  on  $\mathcal{D}$  can be induced from t-structures  $(\tilde{\mathcal{D}}'^{\leq 0}, \tilde{\mathcal{D}}'^{\geq 0})$  and  $(\tilde{\mathcal{D}}''^{\leq 0}, \tilde{\mathcal{D}}''^{\geq 0})$  by [2], where

$$\begin{aligned}\text{obj } \tilde{\mathcal{D}}_1^{\leq 0} &= \{(X, e) \in \tilde{\mathcal{D}} \mid \tilde{i}^*((X, e)) \in \tilde{\mathcal{D}}'^{\leq 0} \text{ and } \tilde{j}^!((X, e)) \in \tilde{\mathcal{D}}''^{\leq 0}\}, \\ \text{obj } \tilde{\mathcal{D}}_1^{\geq 0} &= \{(X, e) \in \tilde{\mathcal{D}} \mid \tilde{i}^!((X, e)) \in \tilde{\mathcal{D}}'^{\geq 0} \text{ and } \tilde{j}^*((X, e)) \in \tilde{\mathcal{D}}''^{\geq 0}\}.\end{aligned}$$

It is straightforward to see that  $(\tilde{\mathcal{D}}_1^{\leq 0}, \tilde{\mathcal{D}}_1^{\geq 0}) = (\tilde{\mathcal{D}}^{\leq 0}, \tilde{\mathcal{D}}^{\geq 0})$ .

According to Theorems 1 and 2 in [7], we have

**Proposition 17.** *Let  $\mathcal{D}$ ,  $\mathcal{D}'$ ,  $\mathcal{D}''$  be triangulated categories,  $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$  and  $(\mathcal{D}''^{\leq 0}, \mathcal{D}''^{\geq 0})$  be t-structures on  $\mathcal{D}'$  and  $\mathcal{D}''$ . Assume  $\mathcal{D}$  admits a recollement relative to  $\mathcal{D}'$  and  $\mathcal{D}''$  and  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is the t-structure induced in [2]. Then the following conditions are equivalent:*

- (1)  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is bounded or nondegenerate, respectively;
- (2)  $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$  and  $(\mathcal{D}''^{\leq 0}, \mathcal{D}''^{\geq 0})$  are bounded or nondegenerate, respectively;
- (3)  $(\tilde{\mathcal{D}}'^{\leq 0}, \tilde{\mathcal{D}}'^{\geq 0})$  and  $(\tilde{\mathcal{D}}''^{\leq 0}, \tilde{\mathcal{D}}''^{\geq 0})$  are bounded or nondegenerate, respectively;
- (4)  $(\tilde{\mathcal{D}}^{\leq 0}, \tilde{\mathcal{D}}^{\geq 0})$  is bounded or nondegenerate, respectively.

**Proof.** We only prove the boundedness, the other case can be proved similarly. By Theorem 2 in [7] and the above explanation, we see that (1)  $\Leftrightarrow$  (2) and (3)  $\Leftrightarrow$  (4). Next we will show that (1)  $\Leftrightarrow$  (4). Let  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  be bounded. Then, by the main theorem in [5],  $\mathcal{D}$  is idempotent complete. That is  $\tilde{\mathcal{D}} \simeq \mathcal{D}$ . Thus,  $(\tilde{\mathcal{D}}^{\leq 0}, \tilde{\mathcal{D}}^{\geq 0})$  is bounded. Conversely, suppose that  $(\tilde{\mathcal{D}}^{\leq 0}, \tilde{\mathcal{D}}^{\geq 0})$  is bounded, then for any  $X \in \mathcal{D}$ , there exists  $m, n \in \mathbb{Z}$  such that  $(X, \text{Id}_X) \in \tilde{\mathcal{D}}^{\leq m} \cap \tilde{\mathcal{D}}^{\geq n}$ , that is  $X \in \mathcal{D}^{\leq m} \cap \mathcal{D}^{\geq n}$ . Hence,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is bounded. This completes the proof.  $\square$

## Acknowledgments

We would like to thank Professor Yanan Lin for helpful discussions and the referee for his/her valuable suggestions and comments contributed to the final version of this paper.



## References

- [1] P. Balmer, M. Schlichting, Idempotent completion of triangulated categories, *J. Algebra* 236 (2001) 819–834.
- [2] A.A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers, *Astérisque* 100 (1982) 5–172.
- [3] A.I. Bondal, M.M. Kapranov, Representable functors, Serre functors, and reconstructions, *Izv. Akad. Nauk SSSR Ser. Mat.* 53 (6) (1989) 1183–1205, 1337; English transl.: *Math. USSR Izv.* 35 (3) (1990) 519–541.
- [4] X. Chen, Y. Ye, P. Zhang, Algebras of derived dimension zero, *Comm. Algebra*, in press, math.RT/0608377.
- [5] J. Le, X. Chen, Karoubianness of a triangulated category, *J. Algebra* 310 (2007) 452–457.
- [6] B. Parshall, L. Scott, Derived categories, quasi-hereditary algebras and algebraic groups, *Carlton Univ. Math. Notes* 3 (1988) 1–104.
- [7] Z. Wang, Nondegeneration and boundedness of t-structure induced by recollement, *J. Xiamen Univ.* 45 (1) (2006) 10–13.